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On some interconnections between strict monotonicity, globally uniquely solvable, and P properties in semidefinite linear complementarity problems

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Abstract

In the setting of semidefinite linear complementarity problems on $\mathcal{S}^n$, the implications *strict monotonicity* $\Rightarrow$ $\mathbf{P}_2$ $\Rightarrow$ **GUS** $\Rightarrow$ **P** are known. Here, **P** and $\mathbf{P}_2$ properties for a linear transformation $L : \mathcal{S}^n \to \mathcal{S}^n$ are respectively defined by: $X \in \mathcal{S}^n$, $XL(X) = L(X)X \preceq 0 \Rightarrow X = 0$ and $X \succeq 0,\ Y \succeq 0,\ (X - Y)[L(X) - L(Y)](X + Y) \preceq 0 \Rightarrow X = Y$; **GUS** refers to the global unique solvability in semidefinite linear complementarity problems corresponding to L. In this article, we show that the reverse implications hold for any self-adjoint linear transformation, and for normal Lyapunov and Stein transformations. By introducing the concept of a principal subtransformation of a linear transformation, we show that $L : \mathcal{S}^n \to \mathcal{S}^n$ has the $\mathbf{P}_2$-property if and only if for every $n \times n$ real invertible matrix Q, every principal subtransformation of $\widehat{L}$ has the **P**-property where $\widehat{L}(X) := Q^{\mathrm{T}} L(QXQ^{\mathrm{T}})Q$. Based on this, we show that $\mathbf{P}_2$, **GUS**, and **P** properties coincide for the two-sided multiplication transformation.

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1. Introduction

Given a linear transformation $L : \mathscr{S}^n \to \mathscr{S}^n$ and a matrix $Q \in \mathscr{S}^n$, the *semidefinite linear complementarity problem*, SDLCP(L, Q), is to find a matrix X such that

$$X \succeq 0, \quad Y := L(X) + Q \succeq 0, \quad \text{and} \quad \langle X, Y\rangle := \operatorname{trace}(XY) = 0,$$

where $\mathscr{S}^n$ denotes the space of all real symmetric $n \times n$ matrices and $Z \succeq 0$ means that Z belongs to the cone $\mathscr{S}^n_+$ of all symmetric positive semidefinite matrices in $\mathscr{S}^n$.

This problem, studied in [3,8–11,20,21,25], includes the (standard) LCP [5] and the geometric SDLCP of Kojima et al. [15]. Its applications include primal-dual semidefinite linear programs, control theory, linear and bilinear matrix inequalities [4,18]. It is a special case of a cone complementarity problem, which in turn in a special case of a variational inequality problem [13]. Since the cone $\mathscr{S}^n_+$ is nonpolyhedral (standard) LCP results/concepts cannot be routinely generalized to SDLCPs.

In connection with the semidefinite LCP, various concepts, such as the (strict) monotonicity, **GUS** and **P** properties were introduced and studied in the above cited works. We recall [9] that $L : \mathscr{S}^n \to \mathscr{S}^n$ is said to have the

(a) *monotonicity property* (*strict or strong monotonicity property*) if

$$\langle L(X), X\rangle \geqslant 0 \quad (> 0) \quad \text{for } 0 \neq X \in \mathscr{S}^n;$$

(b) *Globally uniquely solvable* (**GUS**)*-property* if for all $Q \in \mathscr{S}^n$, SDLCP(L, Q) has a unique solution;

(c) **P**-*property* if

$$X \in \mathscr{S}^n, \quad XL(X) = L(X)X \preceq 0 \Rightarrow X = 0;$$

(d) $\mathbf{P}_2$-*property* if

$$X \succeq 0, \;\; Y \succeq 0, \quad (X - Y)[L(X) - L(Y)](X + Y) \preceq 0 \Rightarrow X = Y.$$

One can define, see [8], a (stronger) noncommutative version of (c): $X \in \mathscr{S}^n$, $XL(X) + L(X)X \preceq 0 \Rightarrow X = 0$. This property seems harder to analyze and will not be treated here. It has been shown, see [9], that under (c), SDLCP(L, Q) has a solution for all Q and the system $X \succ 0$, $L(X) \succ 0$ has a solution.

The properties (b)–(d) (and the noncommutative version of (c)) are the semidefinite analogs of the following properties of a matrix $M \in R^{n\times n}$ in the (standard) LCP setting [5,9]:

(b′) For all $q \in R^n$, LCP(M, q) has a unique solution,
(c′) $x \in R^n,\ x * Mx \leqslant 0 \Rightarrow x = 0$,
(d′) $x \geqslant 0, y \geqslant 0, (x - y) * [Mx - My] * [x + y] \leqslant 0 \Rightarrow x = y$,

where '$*$' denotes the Hadamard (i.e., componentwise) product. As is well known, these properties are equivalent to M being a **P**-matrix, that is, all principal minors of M are positive.

Motivated by the equivalence of properties (b′)–(d′), we ask how properties (a)–(d) are related to each other. Since the (generally irreversible) implications

$$\text{strict monotonicity } \Rightarrow \mathbf{P}_2 \Rightarrow \mathbf{GUS} \Rightarrow \mathbf{P} \tag{1}$$

are known (see [20] for the first implication and [9] for the last two implications), we are interested in the reverse implications. Before describing our findings, we briefly recall three important transformations studied in the SDLCP literature.

Given a matrix $A \in R^{n\times n}$, the *Lyapunov*, *Stein*, and *two-sided multiplication* transformations on $\mathcal{S}^n$ are defined (respectively) by

$$L_A(X) := AX + XA^{\mathrm{T}}, \quad S_A(X) := X - AXA^{\mathrm{T}}, \quad M_A(X) := AXA^{\mathrm{T}}. \tag{2}$$

The Lyapunov transformation L_A has been extensively studied in the matrix/control/systems theory literature [1,4,6,7]. A celebrated result of Lyapunov [7,17] states that the continuous linear dynamical system $\mathrm{d}x/\mathrm{d}t = -Ax(t)$ is (globally) asymptotically stable if and only if all the eigenvalues of A lie in the positive right-half plane and that the latter condition holds if and only if the system $X \succ 0,\ L_A(X) \succ 0$ has a solution. In [9], it was shown that these conditions are equivalent to the **P**-property of L_A. In [11], the simultaneous stability of a finite number of matrices was studied based on the composition of several Lyapunov transformations and a fixed point map on $\mathcal{S}^n_+$. Recently, in [24], Sun and Sun express the differentiability properties of the projection map $X \mapsto \Pi_{\mathcal{S}^n_+}(X)$ in terms of Lyapunov transformations; these results were used in [19] to discuss the stability/regularity of a solution of a semidefinite (nonlinear) complementarity problem.

Along with the Lyapunov transformation, the Stein transformation S_A has also been extensively studied in the matrix/control/systems theory literature [1,4,6,7]. It is well known that the discrete linear dynamical system $x(k + 1) = Ax(k)$ is (globally) asymptotically stable if and only if all the eigenvalues of A lie in the open unit disk and that the latter condition holds if and only if the system $X \succ 0,\ S_A(X) \succ 0$ has a solution. In [8], it was shown that these conditions are equivalent to the **P**-property of S_A. Since this **P**-property of S_A is equivalent to (spectral radius) $\rho(M_A) < 1$, see [8], we may view $S_A = I - M_A$ as a semidefinite analog of a nonsingular M-matrix (which is a matrix of the form $I - B$ where B is a nonnegative matrix with spectral radius is less than one) and expect interesting properties. (We may note that [2, Theorem 2.3] contains 50 equivalent properties of a nonsingular M-matrix.) Motivated by S_A, a general cone complementarity result was proved in [8] for transformations of the form $I - S$ where S leaves the (given) cone invariant.

The transformation M_A on $\mathcal{S}^n$ leaves the cone $\mathcal{S}^n_+$ invariant; it is analogous to a nonnegative matrix on R^n (which leaves the cone R^n_+ invariant). Similar to the nonnegative matrices in the standard LCP, one may expect interesting complementarity properties for M_A. Some of these properties have been explored in [3,20].

Our findings in this paper are as follows:

First, by generalizing three recent results involving self-adjoint Lyapunov, Stein, and two-sided multiplication transformations [9,10,20], we show that *the reverse implications hold in* (1) *for any self-adjoint transformation and for normal Lyapunov and Stein transformations.* We conclude that $L : \mathcal{S}^n \to \mathcal{S}^n$ is strictly monotone if and only if $L + L^{\mathrm{T}}$ has the **P**-property where L^{T} denotes the adjoint of L.

Second, by introducing the concept of a principal subtransformation of a linear transformation (see Section 5), we show that L *has the* $\mathbf{P}_2$*-property if and only if for each invertible real matrix* Q, *every principal subtransformation of* $\widehat{L}$ *has the* **P**-*property where* $\widehat{L}(X) := Q^{\mathrm{T}}L(QXQ^{\mathrm{T}})Q$. (Since linear automorphisms of the semidefinite cone $\mathcal{S}^n_+$ are given by $X \mapsto QXQ^{\mathrm{T}}$ for some real invertible Q [16,22], one may view this result as the semidefinite analog of the following LCP/matrix theory result: A matrix $M \in R^{n\times n}$ is a **P**-matrix if and only if for each linear automorphism Λ of R^n_+ (this is of the form $x \mapsto EDx$ where E is a permutation matrix and D is a positive diagonal matrix), every principal submatrix of $\Lambda^{\mathrm{T}}M\Lambda$ is a **P**-matrix.) As a byproduct of this characterization, we show that M_A has the $\mathbf{P}_2$-property if and only if A is either positive definite or negative definite, answering a question raised by Parthasarathy et al. [21]. Combining this with a result of Bhimasankaram et al. [3], *we deduce that* $\mathbf{P}_2$, **GUS** *and* **P** *properties are equivalent for* M_A.

2. Preliminaries

For $x, y \in R^n$, the usual inner product is denoted either by $\langle x, y\rangle$ or by $x^{\mathrm{T}}y$; in C^n, the inner product is denoted by $\langle\cdot,\cdot\rangle_C$.

For a matrix $A \in R^{n\times n}$, we recall the following definitions:

(a) The *trace* of A is the sum of all the diagonal elements of A, or equivalently, the sum of all the eigenvalues of A.
(b) A is *positive semidefinite* (*positive definite*) if the usual inner product $\langle Ax, x\rangle \geqslant 0$ (> 0) for all nonzero $x \in R^n$.
(c) A is *positive stable* if every eigenvalue of A has positive real part.
(d) A is *orthogonal* if $AA^{\mathrm{T}} = I = A^{\mathrm{T}}A$ where I is the $n \times n$ identity matrix.
(e) A is *normal* if $AA^{\mathrm{T}} = A^{\mathrm{T}}A$.

Recall that $\mathcal{S}^n_+$ is the cone of symmetric $n \times n$ positive semidefinite matrices in the space $\mathcal{S}^n$ of real symmetric $n \times n$ matrices. We use the notation

$$X \succeq (\succ) 0$$

to say that X is symmetric and positive semidefinite (positive definite); the notation $X \preceq 0$ means that $-X \succeq 0$. For $A, B \in R^{n\times n}$, we define $\langle A, B\rangle := \mathrm{trace}(AB^{\mathrm{T}})$; in particular, for $X, Y \in \mathscr{S}^n$,

$$\langle X, Y\rangle = \mathrm{trace}(XY).$$

For an $n \times n$ matrix A and for an index set $\alpha \subseteq \{1, 2, \dots, n\}$, we write a_{ij} to denote the (i, j)-entry of A and write $A_{\alpha\alpha}$ to denote the submatrix of A consisting of entries a_{ij} with $i, j \in \alpha$.

For $x \in R^n$, $\|x\|$ denotes its Euclidean norm. For $A \in R^{n\times n}$,

$$\|A\|_{\mathrm{op}} := \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} \quad \text{and} \quad \|A\|_F := \left[\sum_{i,j} a_{ij}^2\right]^{1/2}$$

denote, respectively, the operator norm and the Frobenius norm. The spectral radius of $A \in R^{n\times n}$ is denoted by $\rho(A)$.

Consider a linear transformation $L : \mathscr{S}^n \to \mathscr{S}^n$. Then its transpose $L^{\mathrm{T}} : \mathscr{S}^n \to \mathscr{S}^n$ is defined by

$$\langle L(X), Y\rangle = \langle X, L^{\mathrm{T}}(Y)\rangle \quad (X, Y \in \mathscr{S}^n).$$

L is said to be self-adjoint on $\mathscr{S}^n$ if $L = L^{\mathrm{T}}$, and normal if L commutes with L^{T}. For the Lypunov, Stein, and two-sided multiplication transformations, we note that $(L_A)^{\mathrm{T}} = L_{A^{\mathrm{T}}}$, $(S_A)^{\mathrm{T}} = S_{A^{\mathrm{T}}}$, $(M_A)^{\mathrm{T}} = M_{A^{\mathrm{T}}}$ and L_A, S_A, M_A are normal when A is normal.

Now let $[L] : R^k \to R^k$ (with $k = n(n+1)/2$) be a matrix representing L with respect to a complete orthonormal set in $\mathscr{S}^n$. Then L and $[L]$ have the same (real) eigenvalues and determinant; in particular, if every real eigenvalue of L is positive, then its determinant is positive. Moreover, L is self-adjoint if and only if $[L]$ is symmetric, and L is strictly monotone on $\mathscr{S}^n$ if and only if $[L]$ is positive definite. Hence, if L is self-adjoint and has only positive real eigenvalues, then it is strictly monotone. (This also follows from spectral theory.)

3. Self-adjoint and normal P-transformations

We show below that self-adjoint **P** transformations are strictly monotone.

Theorem 1. *Suppose a linear transformation $L : \mathscr{S}^n \to \mathscr{S}^n$ has the* **P***-property. Then all real eigenvalues of L are positive and hence L has a positive determinant. If, in addition, L is self-adjoint, then it is strictly monotone and the reverse implications hold in* (1).

Proof. Suppose λ is a nonpositive real eigenvalue of L. Then there exists a nonzero $X \in \mathscr{S}^n$ such that $L(X) = \lambda X$ and so $L(X)X = XL(X) = \lambda X^2 \preceq 0$. This contradicts the **P**-property of L. Thus all real eigenvalues of L are positive. It follows that the determinant is positive. Now suppose that L is also self-adjoint. Then all its eigenvalues are real and positive. It follows that L is strictly monotone. $\square$

Remark. The second part of the above theorem generalizes earlier results proved for L_A (when A is symmetric) [9, Theorem 9], for S_A (when A is symmetric or skew symmetric) [10], and for M_A (when A is symmetric) [20, Theorem 6].

Motivated by the above result, we ask if normal **P** transformations are strictly monotone. We answer this in the negative by the following example.

Example. Let

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

This A is normal and positive definite. It can be shown (by direct algebraic manipulation) that M_A has the **P**-property. (This also follows from [3, Theorem 17], and from Corollary 7 in Section 5.) Furthermore, it can be easily shown that M_A is normal. Yet, for

$$X = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix},$$

$\langle M_A(X), X\rangle = -12$, that is, M_A is not strictly monotone.

In spite of the above example, we show below that for normal Lyapunov and Stein transformations, **P**-property indeed implies the strict monotonicity property.

Theorem 2. *If L_A is normal, then the reverse implications hold in* (1).

Proof. It is enough to show that the **P**-property implies strict monotonicity. To this end, let L_A have the **P**-property. Since $L_{A^{\mathrm{T}}} = (L_A)^{\mathrm{T}}$, it follows from $L_A \circ L_{A^{\mathrm{T}}} = L_{A^{\mathrm{T}}} \circ L_A$ that A is normal. Now the **P**-property of L_A implies that A is positive stable (cf. [9, Theorem 5]). We claim that A is positive definite. Let $x \in R^n$ be nonzero. Since A is normal, we have a unitary matrix U and a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ in $C^{n\times n}$ so that $A = U^*DU$ (cf. [14, Theorem 2.5.4]). Then

$$\alpha := \langle Ax, x\rangle = \langle Ax, x\rangle_C = \langle Dz, z\rangle_C = \sum_1^n d_i|z_i|^2, \tag{3}$$

where $z := Ux$ is nonzero in C^n. Since $\alpha \in R$, the last expression in (3) should be equal to its real part. But since A is positive stable, we have $\mathrm{Re}(d_i) > 0$, proving $\alpha > 0$.

Now, for any nonzero $X \in \mathcal{S}^n$,

$$\langle X, L_A(X)\rangle = 2\mathrm{trace}(XAX) = 2\sum_{i=1}^n x_i^{\mathrm{T}} A x_i, \tag{4}$$

where x_i is the ith column in X. We see that $\langle X, L_A(X)\rangle > 0$ for all $X \neq 0$ proving the strict monotonicity of L_A. $\square$

Theorem 3. *Let A be an $n \times n$ real matrix. For the Stein transformation S_A, consider the following statements*:

(i) $\|A\|_{\mathrm{op}} < 1$.
(ii) S_A *is strictly monotone*.
(iii) S_A *has the* **GUS**-*property*.
(iv) S_A *has the* **P**-*property, i.e.*, $\rho(A) < 1$.
(a) $\|A\|_{\mathrm{op}} \leqslant 1$.
(b) S_A *is monotone*.
(c) $\rho(A) \leqslant 1$.

Then,

$$\text{(i)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(iv)},$$
$$\text{(a)} \Longrightarrow \text{(b)} \Longrightarrow \text{(c)} \quad \textit{and} \quad \text{(b)} + \text{(iv)} \Longrightarrow \text{(iii)}.$$

Furthermore, when S_A is normal (i.e., when A is normal), reverse implications hold in the above statements.

Note that we have obvious implications: (i) $\Longrightarrow$ (a), (ii) $\Longrightarrow$ (b) and (iv) $\Longrightarrow$ (c).

Proof. To show (i) $\Longrightarrow$ (ii) and (a) $\Longrightarrow$ (b), note that the Cauchy–Schwartz inequality in $R^{n\times n}$ gives, for $X \in \mathscr{S}^n$,

$$\operatorname{trace}(AXA^{\mathrm{T}}X) = \langle AX, XA\rangle \leqslant \|AX\|_F \|XA\|_F.$$

By writing $X = [x_1, \ldots, x_n]$ (where x_i denotes the ith column of X), we get

$$\|AX\|_F = \left[\sum_1^n \|Ax_i\|^2\right]^{1/2} \leqslant \left[\|A\|_{\mathrm{op}}^2 \sum_1^n \|x_i\|^2\right]^{1/2} = \|A\|_{\mathrm{op}}\|X\|_F.$$

Similarly,

$$\|XA\|_F = \|A^{\mathrm{T}}X\|_F \leqslant \|A^{\mathrm{T}}\|_{\mathrm{op}}\|X\|_F = \|A\|_{\mathrm{op}}\|X\|_F.$$

Hence, $\operatorname{trace}(AXA^{\mathrm{T}}X) \leqslant \|A\|_{\mathrm{op}}^2\|X\|_F^2$. From

$$\langle S_A(X), X\rangle = \|X\|_F^2 - \operatorname{trace}(AXA^{\mathrm{T}}X) \geqslant \|X\|_F^2(1 - \|A\|_{\mathrm{op}}^2),$$

we have the implications (i) $\Longrightarrow$ (ii) and (a) $\Longrightarrow$ (b). The implication (ii) $\Longrightarrow$ (iii) follows from [13, Cor. 3.2]. If (iii) holds, then S_A has the **P**-property [9, Theorem 7]. That $\rho(A) < 1$ and the **P**-property of S_A are equivalent is given in [8, Theorem 11]. We thus have (iv).

Now suppose (b) holds. Then, for all $\epsilon > 0$, the transformation

$$\frac{1}{1+\epsilon}(S_A + \epsilon I) = S_{A/\sqrt{1+\epsilon}}$$

is strictly monotone. The implication (ii) $\Longrightarrow$ (iv) gives $\rho(A/\sqrt{1+\epsilon}) < 1$. Taking the limit, we get (b) $\Longrightarrow$ (c).

The statements (b) and (iv), respectively give, the convexity of the solution set for the SDLCP(S_A, Q) for all $Q \in \mathscr{S}^n$ [9, Theorem 6] and the **P**-property of S_A [9, Theorem 7]. Hence the implication (b) + (iv) $\Longrightarrow$ (iii). To see the last statement, suppose that S_A is normal. From $(S_A \circ S_{A^{\mathrm{T}}})(I) = (S_{A^{\mathrm{T}}} \circ S_A)(I)$, we get $(AA^{\mathrm{T}})^2 = (A^{\mathrm{T}}A)^2$. From the uniqueness of square root [27, Theorem 6.4], we get $AA^{\mathrm{T}} = A^{\mathrm{T}}A$ proving that A is normal. Since $\|A\|_{\mathrm{op}}$ is also the norm of A computed over C^n, and $\rho(A) = \|A\|_{\mathrm{op}}$ (because of normality) we get the last statement of the theorem. □

4. Principal subtransformations

Analogous to the definition of a principal submatrix of a matrix, we now formulate the concept of a principal subtransformation of a linear transformation.

Definition 4. Let $L : S^n \to S^n$ be a linear transformation. For any $\alpha \subseteq \{1, 2, \ldots, n\}$, we define a linear transformation $L_{\alpha\alpha} : S^{|\alpha|} \to S^{|\alpha|}$ by

$$L_{\alpha\alpha}(Z) = [L(X)]_{\alpha\alpha} \quad (Z \in S^{|\alpha|}),$$

where, corresponding to $Z \in S^{|\alpha|}$, $X \in S^n$ is the unique matrix such that $X_{\alpha\alpha} = Z$ and $x_{ij} = 0$ for all $(i, j) \notin \alpha \times \alpha$. *We call $L_{\alpha\alpha}$ the principal subtransformation of L corresponding to α.*

It can be easily verified that for the Lyapunov transformation L_A, we have $(L_A)_{\alpha\alpha} = L_{A_{\alpha\alpha}}$. Since the positive stable property of a matrix is not inherited by its principal submatrices, we may infer that the **P**-property of a transformation is not inherited by its principal subtransformations. We make this statement precise in the following example.

Example. Consider the Lyapunov transformation $L = L_A : S^2 \to S^2$ corresponding to

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

so that for any

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix},$$

$$L(X) = AX + XA^{\mathrm{T}} = \begin{bmatrix} -2y & x+y-z \\ x+y-z & 2(y+z) \end{bmatrix}.$$

Then the principal subtransformation of L_A corresponding to $\alpha = \{1\}$ is given by

$$L_{11}(x) = \left[L\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \right) \right]_{11} = 0 \quad (x \in S^1).$$

We note that A is positive stable and positive semidefinite. Hence L_A has the **GUS** and **P** properties. Yet L_{11} does not have the **GUS** and **P** properties.

Remark. Monotonicity and strict monotonicity properties are inherited by principle subtransformations. Only under certain conditions, principal subtransformations inherit the **GUS** and **P** properties. For example, suppose $\alpha = \{1, 2, \ldots, m\}$, and

$$L\left(\begin{bmatrix} X_{\alpha\alpha} & 0 \\ 0 & 0 \end{bmatrix} \right) = L(X) = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

for any $X_{\alpha\alpha}$. Then it can be easily shown (see [12]) that $L_{\alpha\alpha}$ has the **P**-property (**GUS**-property) whenever L has the **P**-property (respectively, **GUS**-property). □

5. The $\mathbf{P}_2$-property

The $\mathbf{P}_2$-property of a linear transformation $L : \mathcal{S}^n \to \mathcal{S}^n$ is introduced in [9] as a sufficient condition for the **GUS**-property. Answering a question raised in [9], namely, whether strict monotonicity is related to the $\mathbf{P}_2$-property, Parthasarathy et al. [20] show that

strict monotonicity $\Rightarrow \mathbf{P}_2$

and that the converse holds for the Lyapunov transformation L_A and (when A is symmetric) for M_A. In a subsequent article, the same authors [21] show that for a 2×2 or a 3×3-matrix A, M_A has the $\mathbf{P}_2$-property if and only if either A is positive definite or negative definite, and provide an example of M_A for which the $\mathbf{P}_2$-property holds but not strict monotonicity. They also raise the question whether their result holds for any $n \times n$ matrix A.

Below, we characterize the $\mathbf{P}_2$-property, and, as a byproduct, show that for any A, M_A has the $\mathbf{P}_2$-property if and only if A is either positive definite or negative definite.

In what follows, for a given invertible $Q \in R^{n\times n}$, we define a linear transformation $\widehat{L}$ on $\mathcal{S}^n$ by

$$\widehat{L}(X) = Q^{\mathrm{T}} L(QXQ^{\mathrm{T}})Q. \tag{5}$$

Theorem 5. *For any linear transformation L on $\mathcal{S}^n$, the following are equivalent*:

(a) *L has the* $\mathbf{P}_2$*-property*.
(b) *For every invertible $Q \in R^{n\times n}$, every principal subtransformation of $\widehat{L}$ has the* **GUS***-property*.
(c) *For every invertible $Q \in R^{n\times n}$, every principal subtransformation of $\widehat{L}$ has the* **P***-property*.

Proof. (a) $\Rightarrow$ (b). Assume that L has the $\mathbf{P}_2$-property and fix an invertible Q and an index set α. First we claim that $\widehat{L}$ defined in (5) has the $\mathbf{P}_2$-property. To see this, suppose

$$V \succeq 0, \quad W \succeq 0 \quad \text{and} \quad (V - W)[\widehat{L}(V) - \widehat{L}(W)](V + W) \preceq 0.$$

Then $Q(V - W)[\widehat{L}(V) - \widehat{L}(W)](V + W)Q^{\mathrm{T}} \preceq 0$. Upon writing $X := QVQ^{\mathrm{T}}$, $Y := QWQ^{\mathrm{T}}$ and using the definition of $\widehat{L}$, we see that $X \succeq 0, Y \succeq 0$, with $(X - Y)[L(X) - L(Y)](X + Y) \preceq 0$. From the $\mathbf{P}_2$-property of L, we get $X = Y$ and (from the invertibility of Q) $V = W$. This proves the $\mathbf{P}_2$-property of $\widehat{L}$.

Now we show that $(\widehat{L})_{\alpha\alpha}$ has the $\mathbf{P}_2$-property. To simplify the notation, let $\Gamma := \widehat{L}$ and let without loss of generality, $\alpha = \{1, 2, \ldots, m\}$, $m \leqslant n$. Suppose $X_m \succeq 0$, $Y_m \succeq 0$ with

$$(X_m - Y_m)[\Gamma_{\alpha\alpha}(X_m) - \Gamma_{\alpha\alpha}(Y_m)](X_m + Y_m) \preceq 0,$$

where X_m and Y_m belong to $\mathscr{S}^m$. Then letting

$$X = \begin{bmatrix} X_m & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} Y_m & 0 \\ 0 & 0 \end{bmatrix}$$

and writing

$$\Gamma(X) - \Gamma(Y) = \begin{bmatrix} \Gamma_{\alpha\alpha}(X_m - Y_m) & * \\ * & * \end{bmatrix},$$

we see that

$$(X - Y)[\Gamma(X) - \Gamma(Y)](X + Y) = \begin{bmatrix} E_m & 0 \\ 0 & 0 \end{bmatrix} \preceq 0,$$

where $E_m := (X_m - Y_m)[\Gamma_{\alpha\alpha}(X_m) - \Gamma_{\alpha\alpha}(Y_m)](X_m + Y_m)$. Now from the $\mathbf{P}_2$-property of Γ, we have $X = Y$ and hence $X_m = Y_m$. This shows that $\Gamma_{\alpha\alpha}$, that is, $(\widehat{L})_{\alpha\alpha}$ has the $\mathbf{P}_2$-property. Since $\mathbf{P}_2$-property implies the **GUS**-property [9, Remark 7], we see that for all invertible Q and all α, $(\widehat{L})_{\alpha\alpha}$ has the **GUS**-property. Thus (b) holds.

(b) $\Rightarrow$ (c). This is immediate since the **GUS**-property implies the **P**-property always.

(c) $\Rightarrow$ (a). To some extent, our proof follows that of Theorem 4 in [20] where it is shown that strict monotonicity implies the $\mathbf{P}_2$-property. Assume that (c) holds and suppose that $X \succeq 0, Y \succeq 0, X \neq Y$ with $(X - Y)[L(X) - L(Y)](X + Y) \preceq 0$. Since $X + Y$ is symmetric, positive semidefinite and nonzero, there exist a (real) invertible matrix Q such that

$$X + Y = Q \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{\mathrm{T}},$$

where I_r is the identity matrix of size $r \times r$ and $1 \leqslant r \leqslant n$ (see [27, Theorem 6.3] or the proof of Theorem 4 in [20]). Put $A := Q^{-1}X(Q^{-1})^{\mathrm{T}}$ and $B := Q^{-1}Y(Q^{-1})^{\mathrm{T}}$. Then A and B are symmetric positive semidefinite matrices with

$$A + B = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that

$$A = \begin{bmatrix} A_r & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_r & 0 \\ 0 & 0 \end{bmatrix},$$

where A_r and B_r are $r \times r$-matrices. Now $(X - Y)[L(X) - L(Y)](X + Y) \preceq 0$ yields

$$Q^{-1}(X - Y)(Q^{-1})^{\mathrm{T}} Q^{\mathrm{T}}[L(QAQ^{\mathrm{T}}) - L(QBQ^{\mathrm{T}})]QQ^{-1}(X + Y)(Q^{-1})^{\mathrm{T}} \preceq 0.$$

This gives

$$(A - B)[\widehat{L}(A) - \widehat{L}(B)](A + B) \preceq 0,$$

where $\widehat{L}(Z) := Q^{\mathrm{T}} L(QZQ^{\mathrm{T}})Q$. Writing

$$\widehat{L}(A) - \widehat{L}(B) = \begin{bmatrix} P & N \\ N^{\mathrm{T}} & R \end{bmatrix},$$

we get from $(A - B)[\widehat{L}(A) - \widehat{L}(B)](A + B) \preceq 0$,

$$\begin{bmatrix} A_r - B_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P & N \\ N^{\mathrm{T}} & R \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (A_r - B_r)P & 0 \\ 0 & 0 \end{bmatrix} \preceq 0.$$

This implies that $(A_r - B_r)P \preceq 0$ (which means that $(A_r - B_r)P$ is symmetric and negative semidefinite). By the block form of $A - B$, we note that $P = (\widehat{L})_{\alpha\alpha}(A_r - B_r)$ where $\alpha = \{1, 2, \ldots, r\}$. Thus

$$(A_r - B_r)(\widehat{L})_{\alpha\alpha}(A_r - B_r) \preceq 0.$$

By our assumption, $(\widehat{L})_{\alpha\alpha}$ has the **P**-property. Hence $A_r = B_r$ proving $A = B$ and $X = Y$. This is a contradiction. Therefore L has the $\mathbf{P}_2$-property. This completes the proof. □

Remark. Suppose L has the $\mathbf{P}_2$-property. Then, because of Lemma 1, for every invertible Q, each principal subtransformation of $\widehat{L}$ has positive determinant. It is not clear if the converse holds.

Corollary 6. *For any* $A \in R^{n \times n}$, *consider* M_A *defined by* (2). *Then the following statements are equivalent*:

(a) *A is positive definite or negative definite.*
(b) *M_A has the* $\mathbf{P}_2$*-property.*

Remark. Bhimasankaram et al. [3] have shown that for M_A, **GUS**, **P**, and $\mathbf{R}_0$-properties are equivalent to (a). Hence for M_A, $\mathbf{P}_2$, **GUS**, and **P** properties are equivalent.

Proof. (a) ⇒ (b). Assume without loss of generality that A is positive definite. We verify condition (c) of the previous theorem. For any invertible Q,

$$\widehat{M_A}(X) = Q^{\mathrm{T}} M_A(QXQ^{\mathrm{T}})Q = Q^{\mathrm{T}} AQXQ^{\mathrm{T}} A^{\mathrm{T}} Q = BXB^{\mathrm{T}} = M_B(X),$$

where $B := Q^{\mathrm{T}}AQ$. Note that B is also positive definite as Q is invertible. We need to show that every principal subtransformation of M_B has the **P**-property. Without loss of generality, let $\alpha = \{1, 2, \ldots, m\}$. Let $0 \neq Z \in S^m$ be such that

$$Z(M_B)_{\alpha\alpha}(Z) = (M_B)_{\alpha\alpha}(Z)Z \preceq 0.$$

Writing B in the block form

$$B = \begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

where E conforms with Z, we get $(M_B)_{\alpha\alpha}(Z) = EZE^{\mathrm{T}}$. Writing $Z = V^{\mathrm{T}}DV$ where V is orthogonal and D is diagonal and nonzero, we get $DM_C(D) \preceq 0$ with $C := VEV^{\mathrm{T}}$ (which is positive definite). By using an appropriate permutation matrix, we may assume that

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where D_1 is diagonal and invertible. Writing C in a block form that conforms with D and letting C_1 be the submatrix that conforms with D_1, we see that $DM_C(D) \preceq 0$ yields $D_1C_1D_1C_1^{\mathrm{T}} \preceq 0$. Note that C_1 is positive definite. For ease of notation, we assume $D_1 = D$ and $C_1 = C$ so that D is invertible and C is positive definite. Let λ be a negative eigenvalue of $DCDC^{\mathrm{T}}$ with a corresponding eigenvector u. From $DCDC^{\mathrm{T}}u = \lambda u$, we get $CDC^{\mathrm{T}}u = \lambda D^{-1}u$. Hence

$$\langle CDC^{\mathrm{T}}u, DC^{\mathrm{T}}u\rangle = \lambda\langle D^{-1}u, DC^{\mathrm{T}}u\rangle = \lambda\langle u, C^{\mathrm{T}}u\rangle,$$

where we compute the inner product in an appropriate Euclidean space. Since C is positive definite, the left-hand side of the above expression is positive while the right-hand side is negative. Thus we reach a contradiction.

The implication (b) $\Rightarrow$ (a) can be proved, as in [21], as follows: Suppose $x^{\mathrm{T}}Ax = 0$ with $x \neq 0$. With $X = xx^{\mathrm{T}}$, $Y = 0$, we have $(X - Y)M_A(X - Y)[X + Y] = 0$, contradicting the $\mathbf{P}_2$-property. □

6. Concluding remarks

In this paper, we studied the reverse implications in the statement

strict monotonicity $\Rightarrow$ $\mathbf{P}_2$ $\Rightarrow$ **GUS** $\Rightarrow$ **P**.

We showed that the reverse implications hold for self-adjoint transformations and for normal Lyapunov and Stein transformations. We also gave a characterization of the $\mathbf{P}_2$-property in terms of principal subtransformations $\widehat{L}$ defined with respect to invertible Qs. We do not know if the same characterization holds when Q is restricted to orthogonal matrices. More open questions can be found in [12] where one also finds an elaborated discussion on topics covered in this article.

References

[1] R. Bellman, Introduction to Matrix Analysis, SIAM, Philadelphia, 1995.

[2] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, SIAM, Philadelphia, 1994.

[3] P. Bhimasankaram, A.L.N. Murthy, G.S.R. Murthy, T. Parthasarathy, Complementarity problems and positive definite matrices, Research Report, Indian Statistical Institute, Street No. 8, Habshiguda, Hyderabad 500 007, India, 2000 (Revised June 27, 2001).

[4] S. Boyd, L.E. Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM, Philadelphia, 1994.

[5] R.W. Cottle, J.-S. Pang, R. Stone, The Linear Complementarity Problem, Academic Press, Boston, 1992.

[6] B.N. Datta, Stability and inertia, Linear Algebra Appl. 302–303 (1999) 563–600.

[7] F.R. Gantmacher, Theory of Matrices, Vol. II, Chelsea Publishing Company, New York, 1959.

[8] M.S. Gowda, T. Parthasarathy, Complementarity forms of the theorems of Lyapunov and Stein, and related results, Linear Algebra Appl. 320 (2000) 131–144.

[9] M.S. Gowda, Y. Song, Errata: On semidefinite linear complementarity problems, Math. Prog. Series A 88 (2000) 575–587.

[10] M.S. Gowda, Y. Song, Errata: On semidefinite linear complementarity problems, Math. Prog. Series A 91 (2001) 199–200.

[11] M.S. Gowda, Y. Song, Some new results for the semidefinite linear complementarity problem, SIAM J. Matrix Anal. 24 (2002) 25–39.

[12] M.S. Gowda, Y. Song, G. Ravindran, Some interconnections between strict monotonicity, GUS, and P properties in semidefinite linear complementarity problems, Research Report TR01–11, Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, October 2001.

[13] P.T. Harker, J.-S. Pang, Finite dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms, and applications, Math. Prog. 48 (1990) 161–220.

[14] R. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.

[15] M. Kojima, S. Shindoh, S. Hara, Interior-point methods for the monotone semidefinite linear complementarity problems, SIAM Optim. 7 (1997) 86–125.

[16] R. Loewy, On ranges of real Lyapunov transformations, Linear Algebra Appl. 13 (1976) 79–89.

[17] M.A. Lyapunov, Probléme général de la stabilité des mouvement, Ann. Fac. Sci. Toulouse 9 (1907) 203–474 (French, translation of the original paper published in 1893 in Comm. Soc. Math. Kharkow; reprinted as vol. 17 in Annals of Mathematics Study, Princeton University Press, Princeton, NJ, 1949).

[18] M. Mesbahi, G.P. Papavassilopoulos, A cone programming approach to the bilinear matrix inequality problem and its geometry, Math. Prog. 77 (1997) 247–272.

[19] J.-S. Pang, D. Sun, J. Sun, Semismooth homeomorphims and strong stability of semidefinite and Lorentz complementarity problems, Research Report, Department of Mathematical Sciences, The Johns Hopkins University, Baltimore, Maryland 21228, USA (revised) September 2002.

[20] T. Parthasarathy, D. Sampangi Raman, B. Sriparna, Relationship between strong monotonicity, $\mathbf{P}_2$-property, and the GUS property in semidefinite LCPs, Math. Operations Res. 27 (2002) 326–331.

[21] T. Parthasarathy, D. Sampangi Raman, B. Sriparna, $\mathbf{P}_2$-property need not imply strong monotonicity in the semidefinite LCPs, Research Report, Indian Statistical Institute, Chennai, June 2001.

[22] H. Schneider, Positive operators and an inertia theorem, Numerische Mathematik (1965) 11–17.

[23] P. Stein, Some general theorems on iterants, J. Res. Natl. Bur. Standards 48 (1952) 82–83.
[24] D. Sun, J. Sun, Semismooth matrix valued functions, Math. Operations Res. 27 (2002) 150–169.
[25] P. Tseng, Merit functions for semi-definite complementarity problems, Math. Prog. 83 (1998) 159–185.
[26] N. Yamashita, M. Fukushima, A new merit function for SDCP, in reformulation: nonsmooth, piecewise smooth, semismooth and smoothing methods, Kluwer Academic Publishers, Boston, 1999.
[27] F. Zhang, Matrix Theory, Springer-Verlag, New York, 1999.